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CHOICE OVER ASSET ECONOMIES: DEFAULT RISK AND CORPORATE LEVERAGE

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This paper attempts to clarify the apparent conflict between the recent contribution of Stiglitz and Smith (S–S) and the established Modigliani–Miller (M–M) leverage theorem. The two approaches differ in their treatment of asset creation. Whereas M–M restrict their discussion to a given set of competitive asset markets, S–S consider the addition of an extra asset to the original systems.

1. Introduction

In 1958 Modigliani and Miller proved a classic theorem in the theory of finance: they showed that in a perfect capital market, in equilibrium, corporate value is invariant to the debt–equity ratio. This proposition (now known as the M–M theorem) introduced a long and often confused debate over the assumptions necessary for the proof. During the debate alternative proofs of the M–M theorem appeared as corollaries of more general systems, but these contributions compounded the confusion, because each model was based upon assumptions peculiar to itself, and it was not clear what crucial assumptions were required for the invariance theorem. Subsequently, Stiglitz (1969) published an important paper surveying the literature, and discussing necessary conditions for the M–M theorem. His paper was important also, because it contained one of the first systematic discussions of the effects of bankruptcy on the leverage issue. In a later paper, Stiglitz (1972), went further: 'Previous studies have shown that under very general conditions, if there is no chance of bankruptcy, then financial policy has no effect on the value of the firm; there is no optimal debt–equity ratio. Under certain very restrictive conditions, the no bankruptcy condition may be removed. We show that when these restrictive conditions are not satisfied, and when there is a real possibility of bankruptcy, if the firm issues too much debt, the firm's valuation will depend on its debt–equity ratio . . . .'[1]

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[1]See Stiglitz (1972, p. 458).

A similar position was taken by Smith (1970, 1972), who asserts the following proposition: 'If a corporation can invest at stochastic constant returns to scale, and the default risk on its bonds is positive, then an investor's optimal portfolio will have the property that he will prefer the corporation to increase, leave unchanged, or decrease its debt–equity ratio according as (corporate leverage is greater than, equal to, or less than private anti-leverage).'[2] Although in a recent paper Ben-Zion and Balch (1973) have shown that Smith has an inconsistency in his paper (i.e., the corporate bond rate is independent of leverage), they have not shown that the removal of the error reintroduces the invariance result. Therefore, it might appear that the Stiglitz–Smith (S–S) results are in direct conflict with the claim that the M–M theorem holds under quite general conditions.[3]

The purpose of this paper is to show that there is no conflict once some of the implicit assumptions made by S–S are made explicit. The central point we wish to make is that whereas the M–M theorem is derived from a given commodity space, and associated equilibrium, S–S are comparing a sequence of commodity spaces with associated competitive equilibria. The key to this interpretation is contained in an aside by Stiglitz,[4] where he observed that when the firm issues enough bonds to the point of default, *so that a new security is created*, then the consumption opportunity set of consumer-shareholders is not independent of the leverage decision. Of course, when the commodity space is altered, a new competitive equilibrium is obtained with different security prices. If consumers know the prices in the new equilibrium, they can compare their equilibrium consumption in the old and new equilibria, and choose which one they prefer. Because consumers differ in their preference maps, then, in general, the ranking will not correspond to indifference.

Before we begin the analysis proper, the reader may find it beneficial if we sketch some of the proofs of the M–M theorem given by previous writers. Our purpose is not to reproduce the mechanics of their proofs, but to focus on the important assumptions.

2. The literature

2.1. The M–M proof

Any discussion of the M–M theorem should begin with the original proof. The proof assumes the existence of an equilibrium for firms in a given 'risk-class'. That is, the value of the firm equals the expected return divided by a risk-adjusted discount-rate appropriate for that risk-class. Assuming that shareholders can borrow on the same terms as the levered firm, then they can always

[2]See Smith (1972, p. 71, th. 2).
[3]For the most general statement of the M–M position, we have referred to the standard text, Fama and Miller (1972).
[4]See Stiglitz (1969, p. 790, fn. 12).

undo corporate leverage by private anti-leverage, so that by arbitrage the value of the firm is independent of leverage.

In a recent book, Fama and Miller (1972), F–M have generalized the proof (by omitting the restrictive risk-class assumption) to 'perfect' capital markets. Their definition of a perfect market is crucial if the theorem is to be understood. They define a perfect market in the conventional, competitive sense, with the added proviso that any security issued by a firm has a perfect substitute in existence. (We shall see that it is in this last assumption of a perfect substitute security where S–S and M–M differ.) Given these assumptions, the M–M theorem can be proved quite easily.

2.2. *Arrow–Debreu markets*

A simple, elegant proof of the M–M theorem was found by Hirshleifer (1970) who deduced it from the equilibrium conditions of the Arrow–Debreu (1970, 1959) uncertainty model, where uncertainty is introduced by defining commodities in terms of possible states of the world. In this formulation, one can show that financial decisions are irrelevant: dividend policy and leverage are a matter of indifference to shareholders. Because objective market prices exist for all contingent commodities, production and consumption decisions can be separated so that the producer, by maximizing profit, maximizes the wealth of the consumer-shareholder.[5]

These results are a direct consequence of the assumption that there exists a complete set of Arrow securities. It has been argued that these markets do not exist in reality because of transaction costs, and the problems associated with moral hazard.[6] Without complete markets, one must cast around for a model that introduces uncertainty, and yet produces determinate results. Fortunately, there exist two models that fit these criteria, but they are severely restricted by a tight set of assumptions.

2.3. *The Diamond model*

In 1967 Peter Diamond published an important paper that has influenced much subsequent work on security market equilibrium. Diamond was careful to state explicitly the limitations of his model. He restricted his analysis to a one-commodity, one-period situation, where consumers could trade in a fixed set of securities. The securities represented claims on production processes that experienced technological uncertainty. Assuming no corporate bankruptcy he

[5]This result has come to be known as the Fisher separation theorem. For a discussion and references to the earlier literature see Hirshleifer (1970); and for a general proof in an Arrow–Debreu framework see Milne (1974a).

[6]Arrow (1970) has stressed the moral hazard problem, although there appears to be very little formal analysis of the problem.

showed that corporate value was independent of leverage. But this is a side issue to the more important discovery that for a competitive equilibrium to exist, the production processes had to be decomposable, i.e., firms must be restricted to selling fixed patterns of returns across the states of nature. He observed that when the set of pattern of returns was fixed for the economy, the model was equivalent to the Arrow–Debreu theory, with patterns of returns defined as commodities.

2.4. The mean–variance model

Sharpe (1964) and Lintner (1965) extended the portfolio theory of Tobin–Markowitz to a general equilibrium asset theory. The model assumes that all investors have identical values for the mean and variance of returns for each security, and that investors have utility functions with the mean and variance as arguments. The M–M theorem has been proved using this model. It has been realized that this model is a special case of Diamond's model, where preferences and expectations have been restricted to obtain portfolio separation and the formation of a mutual fund of risky assets.[7]

2.5. The unifying element

In all these models, the claim is made that the market is competitive. It is not hard to show that each model can be considered as isomorphic to the Arrow–Debreu model, by taking an appropriate definition of a commodity. For example, in an asset model of the Diamond type, the commodity is defined as a pattern of returns. The Fama–Miller assumption that a security always has a perfect substitute implies that agents must always choose patterns of returns existing in the market commodity (pattern of returns) space.[8]

At this point we should clear up a confusion that may arise. It is said sometimes that in a perfect capital market, any security created by a firm can be replicated by consumer-shareholders. In existing competitive equilibrium theory, the notion of creating a new commodity cannot be accommodated except in the two following ways: (i) the creation of a commodity can be considered in the trivial sense of allowing agents to take non-zero values for that commodity, given a market price; and (ii) let there be two equilibria – one without the commodity in the commodity space, and one with the extra com-

[7]By portfolio separation is meant that the consumer's choice of assets can be replicated by a risky mutual fund and the safe asset. For necessary and sufficient conditions on consumer preferences to obtain this result, see Cass and Stiglitz (1970). If all consumers agree on the subjective probability distribution, and they are sufficiently alike (for a precise statement, see section 4.5) there exists a market mutual fund of the risky assets.

[8]Alternatively, Ekern and Wilson (1974) have said that any security will be drawn from a set spanned by the available market securities. We consider a spanning set in section 5 below.

modity adjoined. The second equilibrium captures the creation of the new commodity. In proving the M–M theorem, the creation of securities is defined in the first sense, whereas we argue that S–S defined the creation of a new security in the second sense.

3. The model

3.1.

Consider a one-good, one-period world with $i = 1, \ldots, m$ consumers and $j = 1, \ldots, n$ firms. For expository ease, we will limit the analysis initially to two firms, but in a subsequent section we will introduce more firms to obtain a more general result. Because we have restricted ourselves to a single commodity (it could be money with fixed relative prices) and a single period, price uncertainty is absent; and we will assume that uncertainty enters via technological uncertainty. To emphasize the importance of risk aversion, consider the first firm to be safe and the second firm to be risky. Formally, consider the set of states of the world S to be an interval of the real line, i.e., $[\underline{s}, \bar{s}] \subset R$. Because firm 1 has degenerate technological uncertainty then the commodity input I_1 generates a total return rI_1, ($r > 0$, but finite) for all states $s \in S$. Firm 2 suffers from technological uncertainty; but for simplicity assume that the uncertainty is independent of scale, so that for a commodity input of I_2 the total return is $h(s)I_2$, $s \in S$.[9] For ease, consider h to be a monotonically increasing function of s, and $h(\bar{s}) > r$.

To introduce debt and equity financing, assume that the total investment I_2 is the sum of equity E, and debt B, i.e., $I_2 = E+B$. For convenience define a measure of leverage as $\alpha \equiv (B/I_2)$; and also $(1-\alpha) \equiv (E/I_2)$, where $\alpha \in [0, 1]$. Assume that debt pays a nominal physical return rB for those states for which there is no default; and a total return $h(s)I_2$ when default occurs. Equity pays a total return $h(s)I_2 - rB$ for those states where there is no default, and zero when default occurs. The firm defaults on its bonds at the critical state s^*, which is defined as the unique root [if $r > h(\underline{s})$] of

$$h(s)I_2 - rB = I_2(h(s) - r\alpha) = 0.$$

Therefore, we have $s^* = s^*(\alpha)$; and from the assumption that $h(s)$ is monotonically increasing, $s^*(\alpha)$ is single valued.

[9]This is a simplifying assumption that conforms to the M–M notion of a risk-class; and it has been a common enough assumption in the finance literature. Nevertheless Diamond gave a brief discussion of the case where the pattern of returns was not independent of scale, and in turn this has led to a debate on investment rules. For a partial bibliography on the debate, and a general discussion of choice over asset economies, see Milne (1974b). This companion paper includes an example identical to the model used here, except that leverage is fixed and the patterns of returns can be altered by production decisions.

3.2.

Assume that the ith consumer maximizes his von Neumann–Morgenstern utility function $u_i \in C^2$, defined over consumption $x_i(s)$ in the state of the world s, and given his subjective probability distribution function $G_i(s)$. The consumer, therefore, has an objective function (where the integral is a Lebesgue–Stieltjes integral),

$$U_i = \int_{\underline{s}}^{\bar{s}} u_i(x_i(s))\,\mathrm{d}G_i(s), \tag{1}$$

which we will assume exhibits risk aversion, i.e., $u_i' > 0$, $u_i'' < 0$. At this stage we will not place any further restrictions on the objective function.

Assume that the consumer has an initial endowment of commodity $W_i > 0$, and he can purchase proportions θ_{il} of the values of the assets V_l, $l = 1, B, E$. Normalizing the assets in commodity units A_l, and with prices p_l, we have $V_1 = p_1 I_1$, $V_B = P_B B$, $V_E = p_E E$. Let the asset market be competitive so that the ith consumer's budget constraint (assuming non-satiety) is

$$\sum_l \theta_{il} p_l A_l = W_i, \qquad \forall_i. \tag{2}$$

The proportions of the assets purchased entitle the consumer to a proportion of the physical returns; therefore, the consumption set is defined by

$$x_i(s) = \begin{cases} \theta_{i1} I_1 r + \theta_{iB} h(s) I_2, & s \in [\underline{s}, s^*], \\ \theta_{i1} I_1 r + \theta_{iB} r\alpha I_2 + \theta_{iE} I_2 [h(s) - r\alpha], & s \in [s^*, \bar{s}]. \end{cases} \tag{3}$$

Assuming no default,[10] $x_i(s) \geq 0$, and substituting (3) into (1), we obtain

$$\begin{aligned} &\text{Max. } U_i((\theta_{il}) \mid I_1, I_2, \alpha) \\ &= \int_{\underline{s}}^{s^*} u_i(\theta_{i1} I_1 r + \theta_{iB} h(s) I_2)\,\mathrm{d}G_i(s) \\ &\quad + \int_{s^*}^{\bar{s}} u_i(\theta_{i1} I_1 r + \theta_{iB} I_2 \alpha r + \theta_{iE} I_2 [h(s) - \alpha r])\,\mathrm{d}G_i(s), \end{aligned} \tag{4}$$

subject to

$$\sum_l \theta_{il} p_l A_l = W_i.$$

Forming the Lagrangian expression $\mathscr{L}^i$ with λ^i the multiplier associated with the budget constraint, the conditions for an interior maximum are (assuming

[10] The no-default assumption is important in ensuring that the consumer does not issue a security which has an actual pattern of returns different from its nominal pattern of returns. For a complete discussion, including its importance in equilibrium existence proofs, see Milne (1974c).

hereafter that the maximum is achieved other than on points of discontinuity),

$$\int_{\underline{s}}^{s^*} u_i' I_1 r \,\mathrm{d}G_i(s) + \int_{s^*}^{\bar{s}} u_i' I_1 r \,\mathrm{d}G_i(s) + \lambda^i p_1 I_1 = 0, \tag{5}$$

$$\int_{\underline{s}}^{s^*} u_i' h(s) I_2 \,\mathrm{d}G_i(s) + \int_{s^*}^{\bar{s}} u_i' I_2 \alpha r \,\mathrm{d}G_i(s) + \lambda^i p_B I_2 \alpha = 0, \tag{6}$$

$$\int_{s^*}^{\bar{s}} u_i' I_2 [h(s) - \alpha r] \,\mathrm{d}G_i(s) + \lambda^i p_E I_2 (1-\alpha) = 0. \tag{7}$$

Conditions (5), (6) and (7) are easily recognizable as general conditions for the consumer's portfolio problem. Along with the market clearing conditions, and value-maximizing conditions for investment,

$$\sum_i \theta_{il} = 1, \qquad l = I, B, E,$$

$$I_1 + B + E = \sum W_i,$$

$$(\partial/\partial I_j)(V_j - I_j) = 0, \qquad j = 1, 2, \tag{8}$$

they form a system of $3(M+1)+3$ equations. But from Walras' Law we know that

$$\sum_i \sum_l \theta_{il} p_l A_l = I_1 + I_2 = W \equiv \sum W_i,$$

so that we have $3m+5$ independent equations to determine $3m$ portfolio holdings (θ_{il}^*), 3 relative prices (the commodity price is 1), and the investment allocation I_1, I_2.

Now this equilibrium depends upon the parameter $\{\alpha\}$. Because we have assumed that production uncertainty is of the special multiplicative form with respect to scale, then variations in the scale of inputs do not affect the patterns of returns. That is, the investment decisions are consistent with a single asset-commodity space. To see this, observe that for an interior solution to the portfolio problem, the necessary conditions are independent of I_1, I_2. On the other hand, variations in α, if there is default, will result in a change in the patterns of returns available to consumers. Therefore variations in α effectively generate a choice over asset equilibria produced by different patterns of returns.

It is important to notice that we have restricted the actions of consumers to the passive role of not creating their own securities. For simplicity, assume that consumers wish to choose between the asset equilibria generated by the financing arrangements of their agent the firm, and they are far-sighted enough to predict with certainty the price vector associated with each asset equilibrium.[11]

[11]The assumption of perfect foresight of equilibrium asset prices is very strong. But if this assumption is not made, we would need to postulate a much more complicated price uncertainty model, which would complicate the simple point we wish to make.

4. Choice over asset equilibria

4.1.

To reveal consumer i's preferences over the set $\alpha \in [0, 1]$, consider the problem

$$\operatorname*{Max.}_{\{\alpha\}} U_i(\theta_{il}(\alpha), I_j(\alpha), \alpha)$$

subject to

$$\sum_l \theta_{il}(\alpha) A_l(\alpha) p_l(\alpha) = W_i, \tag{9}$$

where $(\theta_{il}(\alpha), I_j(\alpha), p_l(\alpha))$ is an equilibrium vector corresponding to the asset economy generated by the commodity space associated with $\alpha \in [0, 1]$. The condition for an interior maximum is

$$\frac{\mathrm{d}\mathscr{L}^i}{\mathrm{d}\alpha} = \sum_l \left[\frac{\partial \mathscr{L}^i}{\partial \theta_{il}} \frac{\mathrm{d}\theta_{il}}{\mathrm{d}\alpha} + \frac{\partial \mathscr{L}^i}{\partial p_l} \frac{\mathrm{d}p_l}{\mathrm{d}\alpha} \right] + \sum_j \frac{\partial \mathscr{L}^i}{\partial I_j} \frac{\mathrm{d}I_j}{\mathrm{d}\alpha} + \frac{\partial \mathscr{L}^i}{\partial \alpha} = 0. \tag{10}$$

Now from the portfolio conditions (5)–(7) it is clear that $\partial \mathscr{L}^i / \partial \theta_{il} = 0$, and also it is easy to show that $\partial \mathscr{L}^i / \partial I_j = 0$, therefore we find

$$\frac{\mathrm{d}\mathscr{L}^i}{\mathrm{d}\alpha} = I_2[\theta_{iB} - \theta_{iE}] \int_{s^*}^{\bar{s}} u_i' r \,\mathrm{d}G_i(s) \tag{11}$$

$$+ \lambda^i [\theta_{iB} p_B - \theta_{iE} p_E] I_2 + \lambda^i \sum_l \theta_{il} A_l \frac{\mathrm{d}p_l}{\mathrm{d}\alpha} = 0.$$

Substituting from (6) into (11),

$$[\theta_{iE} - \theta_{iB}] \int_{\underline{s}}^{s^*} u_i'(h(s)/\alpha) \,\mathrm{d}G_i(s) + \lambda^i \theta_{iE}(p_B - p_E)$$

$$+ (\lambda^i / I_2) \sum_l \theta_{il} A_l \frac{\mathrm{d}p_l}{\mathrm{d}\alpha} = 0. \tag{11'}$$

The first term of (11′) can be thought of as the gain in utility from a change in the pattern of returns independently of wealth, and the last two terms are wealth effects. (Notice that λ^i is the negative of the marginal utility of wealth.) Eliminating λ^i by (5) we obtain a further variant of (11),

$$\frac{[\theta_{iE} - \theta_{iB}] \int_{\underline{s}}^{s^*} u_i'(h(s)/\alpha) \,\mathrm{d}G_i(s)}{\int_{\underline{s}}^{s^*} u_i' r \,\mathrm{d}G_i(s) + \int_{s^*}^{\bar{s}} u_i' r \,\mathrm{d}G_i(s)} + \theta_{iE}(p_B - p_E)/p_1$$

$$+ (I_2 p_1)^{-1} \sum_l \theta_{il} A_l \frac{\mathrm{d}p_l}{\mathrm{d}\alpha} = 0. \tag{11''}$$

Although (11″) adds little to the previous conditions it will provide a useful comparison with the conditions derived below for the optimal choice of α. Now consider cases which will satisfy (11).

4.2. Shareholder and bondholder: Leverage-preference

Because we have assumed very weak restrictions upon preferences and expectations of consumers, there is no obvious reason why the leverage chosen by consumer i', $\alpha_{i'}$, should correspond with that chosen by consumer i'', $\alpha_{i''}$ ($i' \neq i''$). This conflict cannot be resolved in the existing market set-up, but it is possible that extra market arrangements may achieve a determinate result (see section 5 below). It is in this sense that S–S argue that shareholders are not indifferent to leverage. Notice that Smith's results correspond to the first term of (11′), but he has omitted any effects due to changes in asset prices or, equivalently, in rates of return.

The condition (11) is quite compatible with solutions for portfolio holdings, where a consumer may be a bondholder, but not a stockholder, or vice versa. Stiglitz (1972), by assuming risk-neutral consumers, and differences in expectations, obtains this division between stockholders and bondholders. Needless to say, the argument is not changed in any essential way.

Although we have shown the conflict is possible over the choice of the asset economy generated by the leverage decision, there are several interesting cases where all investors are indifferent to leverage – that is, where the M–M result is obtained.

4.3. No default

It should be clear from our discussion above that if there is no default by the corporate bond, then the corporate bond and the safe asset are perfect substitutes, and in equilibrium they have the same price. To see this, consider an interior solution for the consumer's portfolio problem, i.e., conditions (5)–(7). No default implies that the interval $[\underline{s}, s^*]$ is empty because $h(\underline{s}) \geq \alpha r$.

Therefore conditions (5) and (6) collapse to

$$\int_{\underline{s}}^{\bar{s}} u_i' r \,\mathrm{d}G_i(s) + \lambda^i p_1 = 0, \tag{12}$$

and

$$\int_{\underline{s}}^{\bar{s}} u_i' \alpha r \,\mathrm{d}G_i(s) + \lambda^i p_B \alpha = 0, \tag{13}$$

which imply that $p_1 = p_B$. Furthermore, we have from (7) that

$$\int_{\underline{s}}^{\bar{s}} u_i'(h(s) - \alpha r) \,\mathrm{d}G_i(s) + \lambda^i p_E(1-\alpha) = 0. \tag{14}$$

Now for $r < h(\bar{s})$ it is easy to show that for consumer i to hold positive amounts of equity, $\lim_{\alpha \to 1} p_E = +\infty$. This extreme result depends upon the assumption that the ith consumer is 'small' in his holdings of assets, compared with the aggregate asset holdings. (Because he holds an ϵ-proportion of the total input to the risky asset and reaps a finite return over some states, his physical rate of return goes to infinity.) Of course, this behaviour is limiting behaviour and we will exclude the point $\alpha = 1$ for the next few sections. (A similar argument holds for $\alpha = 0$.)

If $r \leq h(\underline{s})$, then default is absent from our discussion. Alternatively consider $r > h(\underline{s})$ so that default occurs for some $\alpha^* \in (0, 1)$. With α restricted to the no-default interval $(0, \alpha^*)$, then (12), (13), (14) continue to hold.

By summing (13) and (14), consider

$$\int_{\underline{s}}^{\bar{s}} u_i' h(s)\, \mathrm{d}G_i(s) + \lambda^i p_2 = 0,$$

where

$$p_2 \equiv \alpha p_1 + (1-\alpha) p_E.$$

The price p_2 is the price of the (commodity) pattern of returns $h(s)$, which is independent of α, and because p_1 is also independent of α, we obtain

$$\left(\int_{\underline{s}}^{\bar{s}} u_i' h(s)\, \mathrm{d}G_i(s)\right) \Big/ \left(\int_{\underline{s}}^{\bar{s}} u_i' r\, \mathrm{d}G_i(s)\right) = \frac{p_2}{p_1}, \tag{15}$$

which is completely analogous to the usual marginal rate of substitution equated to the price ratio condition, from elementary price theory. We have also proved the M–M theorem, because the value of the corporation is independent of $\alpha \in (0, \alpha^*]$.

In this simple case, we can check by examining the leverage preference condition (11'). Now no default for the corporate bond implies the emptiness of the interval $[\underline{s}, s^*]$, and therefore the integral term vanishes. The remaining two terms can be simplified by observing that $p_B = p_1$ is independent of leverage, whereas $p_E = (p_2 - \alpha p_1)/(1-\alpha)$. Thus

$$\theta_{iE}(p_1 - p_E) + \theta_{iE}(1-\alpha)(p_2 - p_1)/(1-\alpha)^2 = 0, \tag{16}$$

or

$$\theta_{iE}(p_1 - p_E) - \theta_{iE}(p_1 - p_E) \equiv 0.$$

Therefore $\mathrm{d}\mathscr{L}^i/\mathrm{d}\alpha$ is identically zero, when there is no default, implying that consumer i is indifferent to leverage. Because the argument holds for all consumers, then the M–M theorem follows. This proof reveals the importance of the assumption that there is a perfect substitute asset for the corporate bond. If leverage was pushed high enough to force default, so that the corporate bond was not a perfect substitute for the riskless asset, then the proof fails. But if default occurs, *and* the risky bond has a perfect substitute (i.e., there is another security in existence) then again the M–M proof holds. This latter proof will be considered in section 4.

4.4. *Identical investors*

Another special case that will give the M–M result is the situation where all investors are identical. That is, they have identical expectations, utility and wealth. The proof of this proposition is straightforward. Writing $u_i = u$, $G_i(s) = G(s)$, and $W_i = (W/m)$, the portfolio conditions (5)–(7), along with market clearing (8), imply that $\theta_{il} = \theta_l = \theta$, i.e., all consumers hold the same proportion θ of each asset l. For the choice of α, the condition (11′) becomes (observing that p_1 and p_2 are independent of α),

$$(\theta-\theta)\int_{\underline{s}}^{s^*} u'(h(s)/\alpha)\,\mathrm{d}G(s)+\lambda\theta(p_B-p_E)-\lambda\theta(p_B-p_E)\equiv 0. \qquad (17)$$

Therefore $\mathrm{d}\mathscr{L}^i/\mathrm{d}\alpha$ is identically zero, and the replicated consumer is indifferent to leverage. The result is quite trivial because the set of consumers acts as a single consumer who must hold all the assets. Therefore the representative consumer must hold bonds and equity in the same proportions as issued by the risky firm; and that implies that the market pattern of returns always conforms to the original production patterns.

4.5. *Mutual funds and leverage*

We have just seen that if all consumers are identical, then the M–M theorem holds, because all assets are held by all consumers in the same proportion. We observed in the introductory survey, section 1, that the mean-variance formulation implied that all investors would hold securities in a single mutual fund, and, of course, that implied consumer indifference to leverage. Thus, if we can find necessary and sufficient conditions for the existence of a single mutual fund, composed of all the risky assets, then we will have isolated an important class where the M–M theorem holds.

In a recent contribution Cass and Stiglitz (1970) have found necessary and sufficient conditions for portfolio separation, i.e., conditions under which the consumer's opportunity set can be reproduced by two mutual funds partitioning the set of assets. A special case of this more general property is 'monetary' separation, where one mutual fund contains all the risky assets and the other is the safe asset. Using the Cass–Stiglitz results on separation, it is not difficult to show that all investors will purchase risky assets in the same proportion if[12] (i) all investors have the same subjective probability distribution over the states of the world, and (ii) all investors have utility functions of the form:

$$u_i(x_i) = -A_i\, e^{-(x_i/A_i)}, \qquad B = 0, \tag{a}$$

$$u_i(x_i) = \ln(A_i + x_i), \qquad B = 1, \tag{b}$$

or

$$u_i(x_i) = [(B-1)/B^2][A_i + Bx_i]^{(B-1)/B}, \qquad B \neq 0,1. \tag{c}$$

and B is a common value for all investors.

Given these conditions, it follows that for each consumer i, $\theta_{iB} = \theta_{iE} = \theta_i$, when the corporate bond is a risky asset. Again, as with the case of the identical consumers, each consumer holds a share θ_i of the risky mutual fund, which has a total return $h(s)$ and a price p_2.[13] Because p_1 and p_2 are independent of α, the condition (11′) is identically zero, and our assertion is proved. (Of course, if the corporate bond is not risky, leverage indifference is proved as in 4.3.)

An alternative way of obtaining this result is to observe the following argument. Consider the equilibrium generated by the riskless pattern of returns and risky market portfolio pattern of returns. Because of the portfolio separation property of preferences, consumers are indifferent to any set of risky patterns of returns that have a vector sum equal to the risky portfolio pattern of returns. Therefore, variations in leverage alter the components of the risky set, but not the vector sum; and thus the competitive equilibrium is not disturbed by variations in α. (The same argument applies to 4.4.)

5. N corporations and perfect substitute assets

5.1.

The special cases we examined in 4.4 and 4.5 require severe restrictions on the preferences and expectations of consumers; in fact, the restrictions are so severe

[12]For a proof see Rubinstein (1974), where it is referred to as the universal portfolio separation theorem. The interested reader is directed also to Mossin (1973) for the special case of a quadratic utility function.

We should point out that portfolio separation can be obtained also if the subjective probability distribution is Pareto–Levy, or normal if the variance is finite. These distributions are not strictly applicable to our problem because the pattern of returns are bounded below by zero, and are therefore not symmetric. Furthermore, for the formulation of a market mutual fund of risky assets, the consumers must agree upon the probability distributions across the states.

[13]The similarity in the proofs between identical consumers and the mutual fund case follows because for the mutual fund to be formed, consumers must be identical in their pattern of demands for risky assets. Or, in the space of risky asset preferences, all consumers have identical homothetic preferences, except for increasing transformations.

they cannot be considered particularly realistic, especially when they imply the formation of a single mutual fund, or at least a single mutual fund of risky assets. On the other hand, the M–M theorem has been asserted to exist under quite general conditions – independently of consumer preference restrictions. We can prove that assertion if there exists a perfect substitute asset, or a perfect substitute portfolio, for the risky bond issued by the corporation. There are two propositions we can prove using the perfect substitute approach: (a) the first proposition is a restricted invariance result which includes the analysis of section 4.3 as a special case; (b) the second proposition is a global univalence theorem which corresponds to the conventional statement of the M–M theorem (see F–M).

5.2.

We will consider the first proposition. Let there exist $n > 2$ firms, where the first firm remains as our riskless producer, but the remaining firms $j = 2, \ldots, n$ are risky. In particular, let the pattern of returns for the second firm be described as above, and for firms $j = 3, \ldots, n$ summarize their output patterns as $r_j(s)I_j$, where I_j is the input of physical commodity into the jth firm. We will find it necessary to consider a finite-dimensional state space $s \in \{\underline{s}, \ldots, \bar{s}\} \equiv S$. Assume that the available patterns of returns $r, h(s), r_3(s), \ldots, r_n(s)$ are independent, so that the matrix of available patterns Z has rank $\bar{n} \leqq \underline{s}$, where $\bar{n} \leq n$; consider the finite-dimensional analogue of problem (4),

$$\begin{aligned}
&\text{Max. } U_i(\theta_{il} \mid I_j, \alpha) \\
&= \sum_{S'} u_i(\theta_{i1} I_1 r + \theta_{iB} h(s) I_2 + \sum_{l=3}^{\bar{n}} \theta_{il} r_l(s) I_l)\pi_{is} \\
&\quad + \sum_{S''} u_i(\theta_{i1} I_1 r + \theta_{iB} I_2 \alpha r + \theta_{iE} I_2 [h(s) - \alpha r] \\
&\quad + \sum_{l=3}^{\bar{n}} \theta_l r(s) I)\pi_{is},
\end{aligned} \tag{18}$$

subject to

$$\sum_{l} \theta_{il} p_l A_l = W_i, \qquad l = 1, B, E, 3, \ldots, \bar{n}.$$

Define the sets $S' = \{\underline{s}, \ldots, s_k\}$ and $S'' = \{s_{k+1}, \ldots, \bar{s}\}$, where s_k, s_{k+1} are chosen such that $h(s_k) - \alpha r < 0$, and $h(s_{k+1}) - \alpha r \geq 0$. Because h is strictly increasing in s_k, the sets are well-defined and partition S. The analogues of the portfolio equations (5)–(7) are

$$\sum_{S'} u_i' I_1 r \pi_{is} + \sum_{S''} u_i' I_1 r \pi_{is} + \lambda^i p_1 I_1 = 0, \tag{19}$$

$$\sum_{S'} u_i' h(s) I_2 \pi_{is} + \sum_{S''} u_i' I_2 \alpha r \pi_{is} + \lambda^i p_B I_2 \alpha = 0, \tag{20}$$

$$\sum_{S''} u_i' I_2 [h(s) - \alpha r] \pi_{is} + \lambda^i p_E I_2 (1-\alpha) = 0, \tag{21}$$

$$\sum_{S'} u_i' I_j r_j(s) \pi_{is} + \sum_{S''} u_i' I_j r_j(s) \pi_{is} + \lambda^i p_j I_j = 0, \qquad j = 3, \ldots, n. \tag{22}$$

To reveal consumer i's preferences over the set $\alpha \in [0, 1]$, consider the analogue of condition (11),

$$\frac{\mathrm{d}\mathscr{L}^i}{\mathrm{d}\alpha} = I_2[\theta_{iB} - \theta_{iE}] \sum_{S''} u_i' r \pi_{is} + \lambda^i I_2 [\theta_{iB} p_B - \theta_{iE} P_E]$$

$$+ \lambda^i \sum_{l} \theta_{il} A_l \frac{\mathrm{d}p_l}{\mathrm{d}\alpha} = 0. \tag{23}$$

It is easy to show that the results obtained, when the number of assets was restricted to $l = 1, B, E$, continue to hold for the expanded set of assets. Nevertheless, we can prove a generalization of the result discussed in section 4.3, i.e., when the corporate bond is riskless. Indifference to leverage in the no-default region was shown because the corporate bond had a perfect substitute – the riskless asset. But if the corporation's risky bond, or equity, has a perfect substitute, then the argument is perfectly analogous.

Consider the case where the corporate leverage is restricted to the range $\alpha \in [h(s_k)/r, h(s_{k+1})/r]$, so that S' and S'' are invariant to changes in leverage. Let there exist two assets j_1 and j_2 with the pattern of returns,

$$r_{j1} = \begin{cases} h(s) & \text{for } s \in S', \\ 0 & \text{for } s \in S'', \end{cases}$$

$$r_{j2} = \begin{cases} 0 & \text{for } s \in S', \\ r & \text{for } s \in S'', \end{cases}$$

and competitive prices p_{j1} and p_{j2}, respectively. The portfolio conditions for an interior solution for these assets are

$$\sum_{S'} u_i' h(s) \pi_{is} + \lambda^i p_{j1} = 0, \tag{24}$$

and

$$\sum_{S''} u_i' r\pi_{is} + \lambda^i p_{j2} = 0. \tag{25}$$

But from (20), and by summing (24) and (25) multiplied by α, we find

$$\sum_{S'} u_i' h(s)\pi_{is} + \sum_{S''} u_i' \alpha r \pi_{is} = -\lambda^i(p_{j1} + \alpha p_{j2}) = -\lambda^i p_B \alpha. \tag{26}$$

Therefore

$$p_B = (p_{j1}/\alpha) + p_{j2}.$$

Similarly, by adding (20) and (21),

$$\sum_{S'} u_i' h(s)\pi_{is} + \sum_{S''} u_i' h(s)\pi_{is} = -\lambda^i(\alpha p_B + (1-\alpha)p_E) \equiv \lambda^i p_2. \tag{27}$$

Observe that p_2 is independent of α, because the left-hand side of (27) is independent of α, [λ^i is independent of α by (19)]. Therefore, we have shown the M–M theorem for the restricted range of leverage for which a perfect substitute combination of assets exists. Notice that the corporate asset has a value composed of a linear combination of composite asset prices, i.e.,

$$p_2 = p_{j1} + \alpha p_{j2} + (1-\alpha)p_E.$$

For completeness, it is easy to substitute into condition (23) and show that $\mathrm{d}\mathscr{L}^i/\mathrm{d}\alpha$ is identically zero for restricted leverage.

If there exists a continuous chain of substitutes for the risky bond, over the full range of leverage $\alpha \in [0, 1]$, then the global invariance result can be obtained by linking a sequence of restricted invariance results. There is one important instance where this occurs – a market with a 'spanning set' of securites. A special case of a 'spanning set' is Arrow–Debreu securities.

5.3.

To introduce Arrow–Debreu securites into our model, let there be $\bar{s}$ securities, one for each state, such that the return for the kth A–D security is

$$r_k = \begin{cases} 1 & \text{for } s = s_k, \\ 0 & \text{for } \{s_1, \dots, s_{k-1}, s_{k+1}, \dots, \bar{s}\}, \end{cases}$$

where

$$k \in \{1, \dots, \bar{s}\} \equiv S,$$

and the competitive price is p_k. From an interior solution of the portfolio conditions (19)–(22), it follows that

$$p_1 = r\sum_{S} p_k, \tag{a}$$

$$p_B = \Big(\sum_{S'} h(s_k)p_k\Big)\Big/\alpha + \sum_{S''} p_k r, \tag{b}$$

$$p_E = \sum_{S''} [h(s_k) - r\alpha]p_k/(1-\alpha). \tag{c}$$

Therefore, it follows from (b) and (c) that

$$p_B + p_E = \sum_{S} h(s_k)p_k \equiv p_2,$$

which proves the global M–M theorem, as shown by Stiglitz (1969) and others.[14]

But the set of Arrow–Debreu securities is just one of many sets of securites that span the positive orthant (i.e., the consumption set). Indeed, it is well-known,[15] that with short-sales, a spanning set of securities can be converted into A–D securites by an appropriate choice of weights. Therefore we can generalize the M–M proof above, to spanning sets of securities, by prefacing the proof with the remark that a spanning set can be converted into A–D securities. Clearly this spanning set of securities is what F–M define as a 'perfect' capital market.

5.4. The all-debt firm

In section 4.3 above we excluded the extreme points $\alpha = 0, 1$, corresponding to a complete equity and complete debt firm. Now it is sometimes mentioned in asides in the literature that an all-debt firm with default risk is equivalent to an all-equity firm. In our formulation, this can be shown easily enough in the following way. By market clearance it follows that for $\alpha = 0$, we must have $\theta_{iB} = 0$; therefore the objective function of (18) becomes

$$\sum_{S} u_i\Big(\theta_{i1}I_1 r + \theta_{iE}I_2 h(s) + \sum_{j=3}^{n} \theta_{ij}r_j(s)I_j\Big)\pi_{is}. \tag{18'}$$

[14]It is easy to show that p_E and p_B are non-trivial functions of α, i.e.,

$$\mathrm{d}p_B/\mathrm{d}\alpha = -\Big(\sum_{s'} p_k h(s_k)\Big)\Big/\alpha^2 < 0, \qquad \text{a.e.,}$$

$$\mathrm{d}p_E/\mathrm{d}\alpha = \Big(\sum_{s''} p_k h(s_k) - r\Big)\Big/(1-\alpha)^2 > 0, \qquad \text{a.e.}$$

Observe that the nominal rate of interest (r/p_B) is an increasing function of α when there is default.

[15]See Cass and Stiglitz (1970).

Similarly, if $\alpha = 1$ the objective function becomes

$$\sum_{S} u_i(\theta_{i1} I_1 r + \theta_{iB} I_2 h(s) + \sum_{j=3}^{\bar{n}} \theta_{ij} r_j(s) I_j) \pi_{is}, \tag{18''}$$

as long as $r \geq h(\bar{s})$. This assumption is required to ensure that as leverage becomes complete, the risky bond mirrors the corporate pattern of returns. Clearly (18′) and (18″) are identical and lead to the conclusion that all-debt and all-equity firms are formally identical in this type of model. Indeed, the whole notion of a nominal interest rate (r/p_B) is irrelevant to the analysis. If $r < h(\bar{s})$, then we must restrict our discussion to leverage $\alpha < 1$, because the problem is not well-defined at $\alpha = 1$ (although we can discuss limiting behaviour).

6. Optimal leverage

6.1.

In the previous section we discussed cases where the M–M theorem may, or may not, be applicable. Now we propose to investigate the determinates of the optimal choice of leverage. We will show that when there is a choice over asset spaces generated by variations in leverage, the choice of leverage depends upon the preferences and expectations of all holders of the corporation's securities. Indeed, the public good nature of the creation of a new security market will become obvious when our optimal conditions are compared to the sort of conditions produced by public good problems.

6.2.

The Pareto conditions are derived from the following problem.

$$\begin{aligned} \text{Max. } U_1 = &\sum_{S'} u_1(\theta_{11} I_1 r + \theta_{1B} h(s) I_2 + \sum_{l=3}^{\bar{n}} \theta_{1l} r_l(s) I_l) \pi_{1s} \\ &+ \sum_{S''} u_1(\theta_{11} I_1 r + \theta_{1B} I_2 \alpha r + \theta_{1E} I_2 [h(s) - \alpha r] \\ &+ \sum_{l=3}^{\bar{n}} \theta_{1l} r_l(s) I_l) \pi_{1s}, \end{aligned} \tag{28}$$

subject to

(a) $\quad U_i = \bar{U}_i, \qquad i = 2, \ldots, m,$

(b) $\quad \sum_{i} \theta_{il} = 1, \qquad l = 1, B, E, 3, \ldots, \bar{n},$

(c) $\quad \sum_{j=1}^{\bar{n}} I_j = I.$

We can form a Langrangian expression, associating multipliers with the constraints. The first-order conditions for an interior (i.e., $x_i(s) \geq 0$) solution imply the following condition for optimal α,

$$\sum_i \left\{ \frac{(\theta_{iB}-\theta_{iE}) \sum_{S'} u_i' h(s)\pi_{is}}{\sum_{S'} u_i' r\pi_{is} + \sum_{S''} u_i' r\pi_{is}} \right\} = 0. \tag{29}$$

We observed in 4.2 that consumers with different preferences, expectations and wealth may not be indifferent to leverage if there was no perfect substitute asset for the risky corporate bond. Although consumers may not be able to satisfy their private choice of leverage, there does exist a Pareto solution which determines an optimal leverage set for all consumers. This is a weak criterion, because if consumers undertake extra-market bargaining, where coalitions may form and reform costlessly, then we cannot deduce to whom most of the benefits will flow in Pareto bargain. On the other hand, if the formation of coalitions involve communication costs, then some consumers may be able to exercise monopoly power. For example, consider an extreme case where one consumer – we can call him the entrepreneur – can exploit other bondholders and shareholders because the cost of forming coalitions against him is too costly compared with the feasible gains from bargaining. The entrepreneur will choose the leverage according to the solution of condition (11) (i.e., his private leverage choice), and attract the anticipated clientele. Of course, as the entrepreneur varies leverage over the feasible range, the other consumers change their portfolios; but this variation does not alter the point that they may have preferences over the set of feasible degrees of leverage.

6.3.

In sections 4 and 5 we discussed cases where variations in leverage left the competitive equilibrium undisturbed; and therefore consumers were indifferent to variations in leverage. It is well-known that a competitive equilibrium achieves a Pareto optimum, so therefore the indifference set of leverage ratios is also an optimum set.

7. Conclusion

In this paper we have tried to reconcile the arguments of M–M and S–S, and show that their arguments are not in conflict, but depend upon different assumptions. The models make different assumptions about the dimensions of the asset space, and differ over the concept of asset creation. For M–M, the creation of an asset implies the taking of a non-zero holding of an asset (commodity), given an associated market price. On the other hand, S–S consider the creation of an asset as an addition to the commodity space, and the creation of a new asset equilibrium.

One of the unsatisfactory features of the S–S type theory is the assumption that asset markets are restricted in dimension, and it is only firms that can increase the dimension – and then only to a limited extent. Implicit in this restriction is a much more complicated theory involving the existence of markets and transaction costs. A satisfactory theory would introduce these elements into the model explicitly, and produce conditions for the existence or non-existence of markets. Furthermore, the theory would need to demonstrate that coalitions of consumers (in our case firms) may be able to 'open' a new asset market, whereas isolated consumers may not. Presumably the argument would require set-up costs which were large compared to individual consumer wealth, but could be surmounted by the coalitions.

Alternatively, the M–M argument presumes that the asset market is of fixed dimension for the decisions of economic agents. It would appear from F–M's discussion of a perfect asset market that transaction costs are not important and the 'creation' of a new asset by a firm can be duplicated by any consumer.[16] If this statement is taken to mean that the firm or any consumer can create any pattern of returns for which there exists a market price (or a derivable market price, because it is a perfect substitute for a convex combination of existing assets) then the asset economy must be equivalent to an Arrow–Debreu economy. A number of writers have objected to the reasonableness of complete Arrow–Debreu markets,[17] although positivists could counter this objection by the argument that a theory should be judged by its testable implications and predictive ability, and not by the realism, or otherwise, of its assumptions.

Our reconciliation of the S–S and M–M arguments also offers a *possible* insight into the implicit theorizing of the more traditional finance literature.[18] The traditional theorists argued that corporate value would increase with increases in leverage from zero to some positive level α', remain constant over some interval $[\alpha', \alpha'']$, and decline for 'unreasonable' levels of leverage $[\alpha'', 1]$. The initial phase was rationalized by the tax-deductability of debt interest payments; and, in general, there appears to be little dispute over this point.[19] Because we have omitted taxes in our discussion, we can take it that the initial interval

[16] See Fama and Miller (1972, p. 155).

[17] For example, see Arrow (1970).

[18] From a careful reading of the traditional position [for a statement see Solomon (1963)], I think it would be fair to say that their conclusions on the relationship between corporate value and leverage were derived from casual empiricism, rather than from any rigorous model. Although their theoretical constructions leave much to be desired, we are more interested in their conclusions, which they thought mirrored the market place. I am not suggesting that this argument is what the traditional theorists had in mind, nor that they would agree with this ex post, but that it is *consistent* with their conclusions.

[19] See Fama and Miller (1972, ch. 4, section III. D).

$[0, \alpha']$ is empty, and corporate value is invariant to leverage up to the 'unreasonable' interval $[\alpha'', 1]$. Now it *might* be possible to rationalize this relationship in the following way. For low levels of leverage, the corporate bond is riskless and is a perfect substitute for the riskless asset (say, the government bond), and even with moderate risk of default the corporate bond has substitutes in the market; but with extreme leverage, the asset patterns – especially the equity patterns – may not have substitutes, because they have 'unreasonable' patterns. By 'unreasonable' would have to be meant patterns of returns that gave positive returns in states that were considered by all consumers to have virtually zero probability of occurrence, so that the gains from opening markets would not cover the set-up costs. Of course, from our general equilibrium treatment, we cannot say anything about corporate value over the sequence of asset economies, nor does it make any sense, because all asset prices are presumed to vary. The use of corporate value as a surrogate for consumer preferences fails because the Fisher theorem is no longer applicable.[20] Nevertheless, the point the early theorists were trying to make about consumer preferences at high levels of leverage might be rationalized by the non-existence of markets for extreme patterns of returns.

Finally, we should mention the factor of 'moral hazard' – the confusion of outcomes generated by states of the world and the actions of agents. The model we have used has technological uncertainty, but does not allow any role for agents altering the pattern of returns by production decisions, or even more crudely, entrepreneurial agents absconding with the funds! As Hirshleifer (1970) has observed, the concept of uncertainty is very precise, and eliminates the possibility of some agents exploiting the ignorance of others. I suspect that one could make a case for the proposition that the incentives for deviating from the contractual production, by the equity-holding entrepreneur, are positively related to leverage, as long as the bondholders have difficulty in distinguishing between the outcome of actions by the entrepreneur and outcomes produced by states of the world.[21] Of course, these assumptions would violate the perfect market assumptions made by M–M.

[20]For the Fisher theorem to apply as the firm introduces new assets, we would require a partial equilibrium analysis, so that all asset values are held constant except the firm's equity and debt. Clearly it is no longer necessary that corporate value is invariant to leverage, but it would require further assumptions to show that the value declined with very high leverage. [Implicitly, we must assume $r < h(\bar{s})$. See section 5.4.]

[21]Unfortunately, there has been very little formal analysis of moral hazard, so that there is a danger of it becoming a catch-all for behaviour that might deviate from the standard market theory. For a simple example of moral hazard in the implementation of production techniques, see Milne (1974b).

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